

Schmidt's Game on Certain Fractals

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Abstract

We construct (α, β) and α -winning sets in the sense of Schmidt's game, played on the support of certain measures (very friendly and awfully friendly measures) and show how to derive the Hausdorff dimension for some.

In particular we prove that if K is the attractor of an irreducible finite family of contracting similarity maps of \mathbb{R}^N satisfying the open set condition then for any countable collection of non-singular affine transformations $\Lambda_i : \mathbb{R}^N \rightarrow \mathbb{R}^N$, $\dim K = \dim K \cap (\cap_{i=1}^{\infty} (\Lambda_i(\mathbf{BA})))$ where \mathbf{BA} is the set of badly approximable vectors in \mathbb{R}^N .

0 Introduction

We shall be using Schmidt's game first introduced by W. M. Schmidt [7] for estimating the dimension of certain sets. Let us first define the set of badly approximable numbers. A vector $\mathbf{x} \in \mathbb{R}^N$ is said to be **badly approximable** if there exists $\delta > 0$ such that for any $\mathbf{p} \in \mathbb{Z}^N$, $q \in \mathbb{N}^+$

$$d(\mathbf{x}, \frac{\mathbf{p}}{q}) \geq \delta q^{-\frac{N+1}{N}}. \quad (0.1)$$

We denote the set of all badly approximable points by \mathbf{BA} . The above mentioned game was used by Schmidt, among other things, to tackle the following questions concerning \mathbf{BA} :

1. If $\{\Lambda_i\}_{i=0}^{\infty}$ is a countable collection of non-singular affine transformations $\Lambda_i : \mathbb{R}^N \rightarrow \mathbb{R}^N$, is $\cap_{i=1}^{\infty} (\Lambda_i(\mathbf{BA})) \neq \emptyset$?
2. If $\cap_{i=1}^{\infty} (\Lambda_i(\mathbf{BA})) \neq \emptyset$, what is $\dim \cap_{i=1}^{\infty} (\Lambda_i(\mathbf{BA}))$?

Schmidt proved not only that the intersection is non empty, but is in fact "large" dimension wise, i.e., is of dimension N .

In recent years similar questions have been posed regarding the intersection of \mathbf{BA} with certain subsets of \mathbb{R}^N . For example, let K be any of the following sets: Cantor's ternary set, Koch's curve, Sierpinski's gasket, or in general, an attractor of an irreducible finite family of contracting similarity maps of \mathbb{R}^N satisfying the open set condition. (This condition due to J. E. Hutchinson [2] is discussed in section 4). One may ask the following questions:

1. Is $K \cap \mathbf{BA} \neq \emptyset$?
2. If $K \cap \mathbf{BA} \neq \emptyset$, what is $\dim K \cap \mathbf{BA}$?

Answers to both of these questions have been independently given in [4] and [5] proving $\dim K \cap \mathbf{BA} = \dim K$ for a large family of sets including those mentioned above.

This paper's aim is to extend these results, utilizing Schmidt's game, by answering the following question: If $\{\Lambda_i\}_{i=0}^\infty$ is a countable collection of non-singular affine transformations $\Lambda_i : \mathbb{R}^N \rightarrow \mathbb{R}^N$, what is $\dim K \cap (\cap_{i=1}^\infty (\Lambda_i(\mathbf{BA})))$?

It turns out that for a large family of sets, including for example those mentioned above, the answer is analogous to Schmidt's result in \mathbb{R}^N ,

$$\dim K \cap (\cap_{i=1}^\infty (\Lambda_i(\mathbf{BA}))) = \dim K. \quad (0.2)$$

(See Corollary 4.4 for a precise formulation). We emphasize that our research closely follows in the footsteps of [3], adapting the definitions there for our needs. Although not originally intended for being a friendly environment for Schmidt's game, it turns out that under some modifications, the support of these measures is indeed pretty hospitable to this game.

Section 1 is devoted to introducing a class of measures, exhibiting a geometric feature material for later discussion. We then proceed in describing certain target sets and proving these to be winning sets (section 2). In section 3 we discuss the possibility of inferring a winning set's Hausdorff dimension and finally we study a concrete example of a measure and a target set (section 4).

Notation

\mathbb{R} , \mathbb{Q} and \mathbb{N} denote the set of real, rational and natural numbers respectively.

\mathbb{R}^+ (\mathbb{Q}^+) is the set of non-negative real (rational) numbers while \mathbb{N}^+ denotes the set of strictly positive integers.

Boldface lower case letters (\mathbf{x} , \mathbf{y} ,...etc.) denote points in \mathbb{R}^N .

The function d is the Euclidean distance function between points. If A and B are any two subsets of \mathbb{R}^N , $d(A, B) = \inf \{d(\mathbf{x}, \mathbf{y}) : \mathbf{x} \in A, \mathbf{y} \in B\}$.

λ_N denotes the Lebesgue measure in \mathbb{R}^N .

If (X, d) is a metric space, $B(x, r)$ will denote a closed ball of radius r centered at x , i.e., $B(x, r) = \{z : d(x, z) \leq r\}$, $\partial B(x, r)$ the boundary of $B(x, r)$, i.e., $\{z : d(z, x) = r\}$ and $\text{int} B(x, r)$ denotes the interior of $B(x, r)$ i.e., $\{z : d(x, z) < r\}$.

An affine hyperplane of \mathbb{R}^N will be denoted by \mathcal{L} while $\mathcal{L}^{(\epsilon)}$ is defined to be the ϵ neighborhood of \mathcal{L} , i.e., $\mathcal{L}^{(\epsilon)} = \{\mathbf{x} \in \mathbb{R}^N : d(\mathbf{x}, \mathcal{L}) \leq \epsilon\}$ where ϵ is a non-negative, possibly zero, real number.

Unless otherwise stated, constants are real, strictly positive numbers.

All measures considered in this paper are assumed to be Borel, locally finite on \mathbb{R}^N .

Whenever discussing a measure we denote its support by $\text{supp}(\mu)$.

In order to avoid unnecessary repetitions, all affine transformations referred to in this paper are assumed to be non-singular.

Following conventional notation, for every $U \subset \mathbb{R}^N$ let

$$|U| = \sup \{d(\mathbf{x}, \mathbf{y}) : x, y \in U\}.$$

If $F \subset \mathbb{R}^N$, $\delta > 0$ and $\{U_i\}$ is a countable or finite collection of sets we say that $\{U_i\}$ is a **δ -cover** of F if

$$F \subset \bigcup_{i=1}^{\infty} U_i \text{ and for every } i \text{ } 0 \leq |U_i| \leq \delta.$$

If $F \subset \mathbb{R}^N$ and $s \geq 0$ then for every $\delta > 0$ we define

$$H_{\delta}^s(F) = \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ is a } \delta\text{-cover of } F \right\}$$

and

$$H^s(F) = \lim_{\delta \rightarrow 0} H_{\delta}^s(F)$$

is the **(s)-Hausdorff measure**.

The **Hausdorff dimension** of a set $F \subset \mathbb{R}^N$ is defined by

$$\dim F = \inf \{s : H^s(F) = 0\} = \sup \{s : H^s(F) = \infty\}.$$

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1 Very friendly measures

Definition 1. Call a measure μ **very friendly** if the following conditions are satisfied:

There exist constants $\alpha' < 1$ and r_0 , such that for every $0 < r \leq r_0$ and for every $\mathbf{x} \in \text{supp}(\mu)$ there exist constants C, D and a such that:

- (i) for any $0 \leq \epsilon \leq r$, and any affine hyperplane \mathcal{L} ,

$$\mu(B(\mathbf{x}, r) \cap \mathcal{L}^{(\epsilon)}) < C\left(\frac{\epsilon}{r}\right)^a \mu(B(\mathbf{x}, r)).$$
- (ii) $\mu(B(\mathbf{x}, \frac{5}{6}r)) > D\mu(B(\mathbf{x}, r)).$
- (iii) $\left(\frac{D}{C}\right)^{\frac{1}{a}} \geq \alpha'.$

As a consequence of Definition 1 we prove the following lemma.

Lemma 1.1. Suppose μ is a very friendly measure with α' and r_0 as in Definition 1.1. Let \mathcal{L} be any affine hyperplane. Then for every $0 < r \leq r_0$, if $0 < \alpha < \frac{1}{12}\alpha'$ and $0 \leq \epsilon_0 < \frac{1}{12}\alpha' r$, we have that for every $\mathbf{x} \in \text{supp}(\mu)$ there exists $\mathbf{x}_0 \in \text{supp}(\mu)$ such that

1. $B(\mathbf{x}_0, \alpha r) \subset B(\mathbf{x}, r)$
2. $d(B(\mathbf{x}_0, \alpha r), \mathcal{L}^{(\epsilon_0)}) > \alpha r.$
3. $d(B(\mathbf{x}_0, \alpha r), \partial B(\mathbf{x}, r)) > \alpha r$

Proof. If $d(\mathbf{x}, \mathcal{L}^{(\epsilon_0)}) > 2\alpha r$ the first two conditions are evidently satisfied by choosing $\mathbf{x}_0 = \mathbf{x}$ while for the third notice that $r - \alpha r > \frac{11}{12}r > 2\alpha r$.

Otherwise let $d(\mathbf{x}, \mathcal{L}^{(\epsilon_0)}) \leq 2\alpha r$.

Let $\delta = 1 - \alpha$, $\epsilon = 5\alpha r + 2\epsilon_0$ and denote by $\mathcal{L}_{\mathbf{x}}$ an affine hyperplane parallel to \mathcal{L} passing through \mathbf{x} . We observe that

$$\delta r - \epsilon = (1 - 6\alpha)r - 2\epsilon_0 > \left(1 - \frac{5}{6}\alpha'\right)r - \frac{1}{6}\alpha' r = (1 - \alpha')r \geq 0 \quad (1.3)$$

$$\mu(B(\mathbf{x}, \delta r)) = \mu(B(\mathbf{x}, (1 - \alpha)r)) \geq \mu\left(B\left(\mathbf{x}, \frac{5}{6}r\right)\right) \geq D\mu(B(\mathbf{x}, r)) \quad (1.4)$$

$$\mu(\mathcal{L}_{\mathbf{x}}^{(\epsilon)} \cap B(\mathbf{x}, r)) \leq C\left(\frac{\epsilon}{r}\right)^a \mu(B(\mathbf{x}, r)) = C\left(5\alpha + \frac{2\epsilon_0}{r}\right)^a \mu(B(\mathbf{x}, r)) \quad (1.5)$$

$$< C\left(\frac{31}{36}\alpha'\right)^a \mu(B(\mathbf{x}, r)) < C(\alpha')^a \mu(B(\mathbf{x}, r)) \leq D\mu(B(\mathbf{x}, r))$$

Consequently $\mu(B(\mathbf{x}, \delta r) - (\mathcal{L}_{\mathbf{x}}^{(\epsilon)} \cap B(\mathbf{x}, r))) > 0$ and so we may choose \mathbf{x}_0 to be any point in $\Xi \cap \text{supp}(\mu)$ where $\Xi = B(\mathbf{x}, \delta r) - \mathcal{L}_{\mathbf{x}}^{(\epsilon)}$.

The first condition is fulfilled by our choice of δ . As for the second condition notice that for any $\mathbf{y} \in \Xi$ we have $d(\mathbf{y}, \mathcal{L}^{(\epsilon_0)}) \geq \epsilon - (2\alpha r + 2\epsilon_0) \geq 3\alpha r$. As $d(\Xi, \partial B(\mathbf{x}, r)) = \frac{1}{6}r > 2\alpha r$ the third condition is satisfied as well. \square

Remark 1. *Although we assume full responsibility for coining the somewhat bizarre names used for describing classes of measures studied in this paper, we plead for extenuating circumstances. One should point the blaming finger (if one must) at D. Kleinbock, E. Lindenstrauss and B. Weiss for opening this particular Pandora box, i.e., naming a class of measures friendly [3].*

2 Friendly Schmidt's game

Let (X, d) be a complete metric space and let $\mathcal{S} \subset X$ be a given set (a target set). **Schmidt's game** [7] is played by two players A and B , each equipped with parameters α and β respectively, $0 < \alpha, \beta < 1$. The game starts with player B choosing $y_0 \in X$ and $r > 0$ hence specifying a closed ball $B_0 = B(y_0, r)$. Player A may now choose any point $x_0 \in X$ provided that $A_0 = B(x_0, \alpha r) \subset B_0$. Next, player B chooses a point $y_1 \in X$ such that $B_1 = B(y_1, (\alpha\beta)r) \subset A_0$. Continuing in the same manner we have a nested sequence of non-empty closed sets $B_0 \supset A_0 \supset B_1 \supset A_1 \supset \dots \supset B_k \supset A_k \dots$ with diameters tending to zero as $k \rightarrow \infty$. As the game is played on a complete metric space, the intersection of these balls is a point $z \in X$. Call player A the winner if $z \in \mathcal{S}$. Otherwise player B is declared winner. A strategy consists of specifications for a player's choices of centers for his balls as a consequence of his opponent's previous moves. If for certain α and β player A has a winning strategy, i.e., a strategy for winning the game regardless of how well player B plays, we say that \mathcal{S} is an **(α, β) -winning set**. If it so happens that α is such that \mathcal{S} is an (α, β) -winning set for all possible β 's, we say that \mathcal{S} is an **α -winning set**. Call a set **winning** if such an α exists.

We define the following (target) set. This definition is a modification of the one given in [5].

Definition 2. Suppose $\Omega \subset \mathbb{R}^N$ and let $\mathcal{U} = \{U_j \subset \mathbb{R}^N : j \in \mathbb{N}\}$ be a family of subsets of \mathbb{R}^N . If $I : \mathbb{N} \rightarrow \mathbb{R}^+$ is an increasing function tending to infinity as j tends to infinity and $\rho : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is such that $\rho(r) \rightarrow 0$ as $r \rightarrow \infty$ and decreasing for large enough r , let

$$\mathbf{Bad}^*(\mathcal{U}, I, \rho, \Omega) = \{\mathbf{x} \in \Omega : \exists \delta > 0 \text{ such that } d(\mathbf{x}, U_j) \geq \delta \rho(I(j)) \forall j \in \mathbb{N}\}.$$

As an immediate consequence of the above definition we get:

Proposition 2.1. For $\Omega \in \mathbb{R}^N$, defining $U_j = \left\{ \frac{\mathbf{p}}{j} : j \in \mathbb{N}^+, \mathbf{p} \in \mathbb{Z}^N \right\}$, $I(j) = j$ and $\rho(I(j)) = j^{-\frac{N+1}{N}}$, we have

$$\mathbf{BA} \cap \Omega = \mathbf{Bad}^*(\mathcal{U}, I, \rho, \Omega)$$

In the following corollary we shall show that under certain assumptions, $\mathbf{Bad}^*(\mathcal{U}, I, \rho, \Omega)$ is an (α, β) -winning set.

Corollary 2.2. Suppose μ is a very friendly measure (with r_0 and α' as in Definition 1). Let $\Omega = \text{supp}(\mu)$ and suppose $F : \mathbb{N} \rightarrow \mathbb{R}^+$ is a continuous increasing function, with $F(k) \rightarrow \infty$ as $k \rightarrow \infty$. Define $F^0 = [0, F(0))$ and $F^k = [F(k-1), F(k))$ for any $k > 0$. Let $\mathcal{U} = \{U_j \subset \mathbb{R}^N : j \in \mathbb{N}\}$ be a family of subsets of \mathbb{R}^N . Suppose $0 < \beta < 1$ and $0 < \alpha < \frac{1}{12}\alpha'$ satisfy:

1. for every $k, l \in \mathbb{N}$, for every $\mathbf{x} \in \text{supp}(\mu)$ and for every $r \leq r_0$,
if $I(j_1), \dots, I(j_l) \in F^k$ then $\left(\bigcup_{i=1}^l U_{j_i} \right) \cap B(\mathbf{x}, (\alpha\beta)^k r) \subset \mathcal{L}$ for some affine hyperplane \mathcal{L} ,
2. for every k , $(\alpha\beta)^k \geq \rho(F(k))$.

Then $\mathbf{Bad}^*(\mathcal{U}, I, \rho, \Omega)$ is an (α, β) -winning set.

Proof. Player A's strategy is to play in an arbitrary manner until the first ball of radius $r_I \leq r_0$ is chosen by player B. Let $k_0 \in \mathbb{N}$ be such that $\beta^{k_0+1}r_0 < r_I \leq \beta^{k_0}r_0$. Set $\delta = (\alpha\beta)^{k_0+1}\beta^{k_0}r_0$ and let $r' = (\alpha\beta)^{k_0}r_I$.

We specify player A's strategy from this point on, i.e., set $k_0 = 0$. At his k th move player A has to choose a point $\mathbf{x} \in \text{supp}(\mu)$ such that $A_k = B(\mathbf{x}, \alpha(\alpha\beta)^{k}r') \subset B_k = B(\mathbf{y}, (\alpha\beta)^k r')$ where $\mathbf{y} \in \text{supp}(\mu)$ is player B's k th choice. Let $\mathcal{U}_j = \bigcup_{i=1}^j U_{j_i}$ where $I(j_1), \dots, I(j_l) \in F^k$.

- (a) If $\mathcal{U}_j \cap B(\mathbf{y}, (\alpha\beta)^k r') = \emptyset$, player A may choose $\mathbf{x} = \mathbf{y}$.

By Lemma 1.1(3)

$$d(\mathcal{U}_j, A_k) > \alpha(\alpha\beta)^k r' \geq \delta(\alpha\beta)^k \geq \delta\rho(F(k)) > \delta\rho(I(j)).$$

- (b) Otherwise suppose $\mathcal{U}_j \cap B(\mathbf{y}, (\alpha\beta)^k r') \neq \emptyset$.

by Lemma 1.1(2) player A can pick a point $\mathbf{x} = \mathbf{x}_k$ such that

$$d(\mathcal{U}_j \cap B(\mathbf{y}, (\alpha\beta)^k r'), A_k) > \alpha(\alpha\beta)^k r' > \delta\rho(I(i)).$$

Furthermore, if $\mathcal{U}_j - B(\mathbf{y}, (\alpha\beta)^k r') \neq \emptyset$ then by Lemma 1.1(3)

$$d(\mathcal{U}_j - B(\mathbf{y}, (\alpha\beta)^k r'), A_k) > \alpha(\alpha\beta)^k r' > \delta\rho(I(i)).$$

□

The following lemma due to W.M. Schmidt [7] (Theorem 2) is material for later considerations.

Theorem 2.3. *The intersection of countably many α -winning sets is α -winning.*

3 Awfully friendly measures

Definition 3. Let (X, d) be a metric space. Given $x \in X$ and real numbers $r > 0$ and $0 < \beta < 1$, denote by $N(\beta, x, r)$ the maximum number of disjoint balls of radius βr centered on X and contained in $B(x, r)$.

Definition 4. A measure μ is said to be **awfully friendly** if it is very friendly and satisfies the extra condition:

- (iv) There exists constants $r_1 \leq 1$, M and δ such that for every $0 < r \leq r_1$, $0 < \beta < 1$ and $\mathbf{x} \in \text{supp}(\mu)$, $N(\beta, \mathbf{x}, r) \geq M\beta^{-\delta}$.

For later reference, call δ the exponent of condition (iv).

Proposition 3.1. *Let μ be an awfully friendly measure. If \mathcal{S} is a winning set on (X, d) where $X = \text{supp}(\mu)$ then $\dim \mathcal{S} \geq \delta$ where δ is the exponent of condition (iv).*

In the course of the proof we shall use the following auxiliary lemma.

Lemma 3.2. *Let \mathcal{H} be a Hilbert space and let $w_0 = 2\sqrt{3} - 1$. For any $r \in \mathbb{R}^+$ let $\mathcal{M} = \{B(x_i, r) : i \in \mathbb{N}, x_i \in \mathcal{H} \text{ and for every } i \neq j, \text{ int}B(x_i, r) \cap \text{int}B(x_j, r) = \emptyset\}$.*

Then for any $r_0 < w_0 r$ and $x \in \mathcal{H}$ the ball $B(x, r_0)$ has a non empty intersection with at most two balls from \mathcal{M} .

Remark 2. *The main ideas used in the proof of proposition 3.1 as well as the auxiliary Lemma 3.2 (Lemma 20 in [7]) are due W. M. Schmidt [7].*

Proof. Let μ be an awfully friendly measure and $\beta \leq (\frac{M}{2})^{\frac{1}{5}}$. Thus $N(\beta, \mathbf{x}, r) \geq 2$ for every $\mathbf{x} \in \text{supp}(\mu)$. In order to estimate the Hausdorff dimension of a winning set \mathcal{S} we consider the game from the loser's point of view, player B . Fix β such that

$$2 \leq N(\beta) = \min \{N(\beta, \mathbf{x}, r) : \mathbf{x} \in \text{supp}(\mu), 0 < r \leq r_1\}.$$

At each stage of the game player B may direct the game to $N(\beta)$ disjoint balls and we restrict his moves to these $N(\beta)$ choices. This gives a parametrization of the sequence of balls chosen by player B . Let B_0 be his initially chosen ball, and for $k \in \mathbb{N}^+$, corresponding to his k th move, let $B_k = B_k(j_1, \dots, j_k)$, with $j_i \in \{0, \dots, N(\beta) - 1\}$ $i = 1, 2, \dots, k$. Notice also that given a sequence of positive integers i_1, i_2, \dots there is a *unique* point $x = x(i_1, i_2, \dots)$ contained in *all* balls $B_k = B_k(j_1, \dots, j_k)$. By considering the $N(\beta)$ ways in which player B may direct the game we consider the function

$$f : \{0, \dots, N(\beta) - 1\}^{\mathbb{N}} \rightarrow \mathcal{S}, (\lambda_k)_{k \in \mathbb{N}} \mapsto \bigcap_{k \in \mathbb{N}} B_k(\lambda_1, \dots, \lambda_k) = \{x(\lambda)\}.$$

As every number in the closed unit interval has at least one expansion in base $N(\beta)$ we map the image of f , $\mathcal{S}^* \subset \mathcal{S}$ onto $[0, 1]$ by

$$g : \mathcal{S}^* \rightarrow [0, 1], x(\lambda) \mapsto 0.\lambda_1\lambda_2\dots$$

In view of lemma 3.2, for $0 < w < w_0$ and $0 < \alpha < 1$ any ball of radius $w(\alpha\beta)^k$ intersects at most two of the balls $B_k(j_1, \dots, j_k)$. Let $\mathcal{C} = \{C_l\}_{l \in \mathbb{N}}$ be a cover of $\mathcal{S} \cap \mathcal{K}$ of balls with radius $\rho(C_l) = \rho_l$. As \mathcal{C} covers \mathcal{S}^* we have that $g(\mathcal{C})$ covers $[0, 1]$. Let $\bar{\lambda}$ denote the outer Lebesgue measure. We have

$$\sum_{l=1}^{\infty} \bar{\lambda}(g(C_l)) \geq \bar{\lambda}\left(\bigcup_{l=1}^{\infty} g(C_l)\right) \geq 1. \quad (3.6)$$

Define integers

$$k_l = [k_l^*] \text{ where } k_l^* = \log_{\alpha\beta}(2w^{-1}\rho_l).$$

Notice that:

$(2w^{-1}\rho_l)^{\frac{\log N(\beta)}{|\log(\alpha\beta)|}} = N(\beta)^{-k_l^*}$ and since $k_l^* < k_l + 1$ we get

$$N(\beta)^{-k_l} < N(\beta)N(\beta)^{-k_l^*} = N(\beta)(2w^{-1}\rho_l)^{\frac{\log N(\beta)}{|\log(\alpha\beta)|}}. \quad (3.7)$$

Assuming without loss of generality that for every l , $\rho_l \leq \frac{w}{2}$, there exists $n_0 \in \mathbb{N}$ such that $\frac{w}{2}(\alpha\beta)^{n_0+1} < \rho_l \leq \frac{w}{2}(\alpha\beta)^{n_0}$. It follows that $k_l = n_0$ and so

$$\rho_l < w(\alpha\beta)^{k_l}. \quad (3.8)$$

(3.11) implies that the ball C_l intersects at most two of the balls $B_l(j_1, \dots, j_{k_l})$. As the length of the interval $g(B_l(j_1, \dots, j_l))$ is $N(\beta)^{-k_l}$ we have $\bar{\lambda}(g(C_l)) \leq 2N(\beta)^{-k_l}$. Combining (3.9) and (3.10),

$$1 \leq \sum_{l=1}^{\infty} \bar{\lambda}(g(C_l)) \leq \sum_{l=1}^{\infty} 2N(\beta)^{-k_l} < 2N(\beta)(2w^{-1})^{\frac{\log(N(\beta))}{|\log(\alpha\beta)|}} \sum_{l=1}^{\infty} \rho_l^{\frac{\log(N(\beta))}{|\log(\alpha\beta)|}}.$$

By definition, $\dim \mathcal{S} \geq \frac{\log(N(\beta))}{|\log(\alpha\beta)|} \geq \frac{\delta|\log C_0\beta|}{|\log \alpha| + |\log \beta|} \rightarrow \delta$ as $\beta \rightarrow 0$. \square

Remark 3. If it so happens that $\delta = \dim(\text{supp}(\mu))$ then obviously $\dim(\mathcal{S}) = \dim(\text{supp}(\mu)) = \delta$.

4 Application to Hutchinson's construction

Before giving an example we prove the following theorem and define Hutchinson's construction.

Theorem 4.1. Let $\Lambda : \mathbb{R}^N \rightarrow \mathbb{R}^N$ be an affine map and denote by \mathcal{A} the $N \times N$ matrix associated with the linear part of Λ . For every $\theta \in (0, 1)$ let $R = \theta^{\frac{-N}{N+1}}$ and for every $k \in \mathbb{N}^+$ let

$$U_k = \left\{ \Lambda\left(\frac{\mathbf{p}}{q}\right) : q \in \mathbb{N}^+, \mathbf{p} \in \mathbb{Z}^N \text{ and } R^{k-1} \leq q < R^k \right\}.$$

Denote by V_N the volume of the N -dimensional unit ball. Then for every $r > 0$ such that $r^N < |\det \mathcal{A}| (N!)^{-1} V_N^{-1} \theta^N$ and for every \mathbf{x} there exists an affine hyperplane \mathcal{L} such that

$$U_k \cap B(\mathbf{x}, \theta^{k-1}r) \subset \mathcal{L}.$$

Proof. Assume the contrary and let $\{V_i\}_{i=0}^N$, $V_i = (v_i^1, \dots, v_i^N)$ be $N+1$ independent points in $U_k \cap B(\mathbf{x}, \theta^{k-1}r)$, i.e., not belonging to any single affine hyperplane. Denote by Δ the N -dimensional simplex subtended by them. By a well known result from calculus we have

$$\lambda_N(\Delta) = (N!)^{-1} |\det L'| > 0, \text{ where } L' = \begin{pmatrix} v_1^1 - v_0^1 & \cdot & \cdot & \cdot & v_1^N - v_0^N \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ v_N^1 - v_0^1 & \cdot & \cdot & \cdot & v_N^N - v_0^N \end{pmatrix}.$$

As $\lambda_N(\Delta) > 0$ we have $\det L' \neq 0$.

Consider now the $(N+1 \times N+1)$ matrix $L = \begin{pmatrix} 1 & v_0^1 & \cdot & \cdot & v_0^N \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & v_N^1 & \cdot & \cdot & v_N^N \end{pmatrix}$.

By repeatedly subtracting the first row from all others we get $\det L = \det L''$ where $L'' = \begin{pmatrix} 1 & v_0^1 & \cdot & \cdot & v_0^N \\ 0 & v_1^1 - v_0^1 & \cdot & \cdot & v_1^N - v_0^N \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & v_N^1 - v_0^1 & \cdot & \cdot & v_N^N - v_0^N \end{pmatrix}$ and so $\det L = \det L'$.

Hence, $\lambda_N(\Delta) = |\det A| (N!)^{-1} |\det L|$ where $L = \begin{pmatrix} 1 & \frac{p_0^1}{q_0} & \cdot & \cdot & \frac{p_N^1}{q_0} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \frac{p_0^N}{q_N} & \cdot & \cdot & \frac{p_N^N}{q_N} \end{pmatrix}$

and $\det L \neq 0$ by our assumption.

Notice also that $q_0 \cdot q_1 \cdot \dots \cdot q_N \cdot L = \begin{pmatrix} q_0 & p_0^1 & \cdot & \cdot & p_N^1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ q_N & p_0^N & \cdot & \cdot & p_N^N \end{pmatrix}$,

and as all entries in $q_0 \cdot q_1 \cdot \dots \cdot q_N \cdot L$ are integers it follows that

$$q_0 q_1 \cdot \dots \cdot q_N \cdot |\det L| \geq 1.$$

And so,

$$\lambda_N(\Delta) = (N!)^{-1} |\det A| |\det L| \geq (N!)^{-1} \frac{|\det \mathcal{A}|}{q_0 \cdot \dots \cdot q_N} > (N!)^{-1} |\det \mathcal{A}| R^{-k(N+1)}. \quad (4.9)$$

But,

$$\lambda_N(B(\mathbf{x}, \theta^{k-1}r)) = (\theta^{k-1}r)^N V_N = \theta^{(k-1)N} r^N V_N < |\det \mathcal{A}| \theta^{kN} (N!)^{-1}, \quad (4.10)$$

$$\theta^{kN} = (\theta^{\frac{-N}{N+1}})^{-k(N+1)} = R^{-k(N+1)}, \quad (4.11)$$

and so

$$\lambda_N(B(\mathbf{x}, \theta^{k-1}r)) \leq |\det \mathcal{A}| (N!)^{-1} R^{-k(N+1)}. \quad (4.12)$$

by our assumption on U_k .

As $\Delta \subset B(\mathbf{x}, \theta^{k-1}r)$, (4.9) contradicts (4.12). \square

Remark 4. The proof of Theorem 4.1 closely follows ideas found in [7] (chapter 7).

Hutchinson's construction

A map $\phi : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a **similarity** if it can be written as

$$\phi(\mathbf{x}) = \rho\Theta(\mathbf{x}) + \mathbf{y},$$

where $\rho \in \mathbb{R}^+$, $\Theta \in O(N, \mathbb{R})$ and $\mathbf{y} \in \mathbb{R}^N$. It is said to be **contracting** if $\rho < 1$. It is known (see [Hu] for a more general statement) that for any finite family ϕ_1, \dots, ϕ_m of contracting similarities there exists a unique nonempty compact set K , called the **attractor** or **limit set** of the family, such that

$$K = \bigcup_{i=1}^m \phi_i(K).$$

Say that ϕ_1, \dots, ϕ_m as above satisfy the **open set condition** if there exists an open subset $U \subset \mathbb{R}^N$ such that

$$\phi_i(U) \subset U \text{ for all } i = 1, \dots, m,$$

and

$$i \neq j \implies \phi_i(U) \cap \phi_j(U) = \emptyset.$$

The family $\{\phi_i\}$ is called **irreducible** if there is no finite collection of proper affine subspaces which is invariant under each ϕ_i . Well-known self-similar sets, like Cantor's ternary set, Koch's curve or Sierpinski's gasket, are all examples of attractors of irreducible families of contracting similarities satisfying the open set condition.

Suppose $\{\phi_i\}_{i=1}^m$ is a family of contracting similarities of \mathbb{R}^N satisfying the open set condition, let K be its attractor, δ the Hausdorff dimension of K , and μ the restriction of the δ -dimensional Hausdorff measure to K .

J. Hutchinson [2] gave a simple formula for calculating δ and proved that $\mu(K)$ is positive and finite. Furthermore, μ satisfies the so called power law:

Theorem 4.2. *There exist real numbers $a, b > 0$ such for every $\mathbf{x} \in K$, $0 < r \leq 1$*

$$ar^\delta \leq \mu(B(\mathbf{x}, r)) \leq br^\delta.$$

Before proving our next theorem we review some definitions from [3] needed for the proof.

Definition 5. *If $D > 0$, and $U \subset \mathbb{R}^N$ is an open set, call μ **D-Federer on U** if for all $\mathbf{x} \in \text{supp}(\mu) \cap U$ one has*

$$\mu(B(\mathbf{x}, 3r)) < D\mu(B(\mathbf{x}, r)) \tag{4.13}$$

whenever $B(\mathbf{x}, 3r) \subset U$.

Definition 6. *A measure μ is **Federer** if μ -a.e. \mathbf{x} there exist a neighborhood U of \mathbf{x} and $D > 0$ such that μ is D -Federer on U .*

Definition 7. *We say that μ is **nonplanar** if $\mu(\mathcal{L}) = 0$ for any affine hyperplane \mathcal{L} of \mathbb{R}^N .*

Definition 8. Given C , $a > 0$ and an open subset U of \mathbb{R}^N we say that μ is **absolutely (C, a) -decaying on U** if for any non-empty open ball $B \subset U$ centered in $\text{supp}(\mu)$, any affine hyperplane $\mathcal{L} \subset \mathbb{R}^N$ and any $\epsilon > 0$ one has

$$\mu(B \cap \mathcal{L}^{(\epsilon)}) \leq C \left(\frac{\epsilon}{r} \right)^a \mu(B) \quad (4.14)$$

where r is the radius of B .

Definition 9. μ will be called **absolutely decaying** if for μ -a.e. \mathbf{y} there exists a neighborhood U of \mathbf{y} and $C, a > 0$ such that μ is absolutely (C, a) -decaying on U .

We shall prove the following:

Theorem 4.3. Let $\{\phi_1, \dots, \phi_k\}$ be a finite irreducible family of contracting similarity maps of \mathbb{R}^N satisfying the open set condition. Let K be its attractor with $\dim K = \delta$. Let μ be the restriction of H^δ to K . Then μ is awfully friendly with δ the exponent of condition D.

Proof. Set $r_0 = 1$. The power law implies that for every $\mathbf{x} \in K$, $0 < r \leq r_0$

$$a\left(\frac{5}{6}\right)^\delta r^\delta \leq \mu(B(\mathbf{x}, \frac{5}{6}r)) \leq b\left(\frac{5}{6}\right)^\delta r^\delta$$

and

$$ar^\delta \leq \mu(B(\mathbf{x}, r_0)) \leq br^\delta.$$

Combining these inequalities we get

$$\mu(B(\mathbf{x}, \frac{5}{6}r)) \geq D\mu(B(\mathbf{x}, r))$$

where $D = \frac{a}{b}\left(\frac{5}{6}\right)^\delta$.

Following [3](Theorem 2.3, Lemma 8.2 and 8.3), there exist C and a such that μ is absolutely (C, a) -decaying on any ball of radius $r = 1$ centered on $\text{supp}(\mu)$.

Using the notation of Definition 1, μ is very friendly with

$$r_0 = 1, \alpha' = \left(\frac{D}{C} \right)^{\frac{1}{a}}. \quad (4.15)$$

We are left to prove condition (iv) of Definition 4.

Let $r \leq 1$, $0 < \beta < 1$ and consider a ball $B(\mathbf{x}, r)$ with $\mathbf{x} \in K$. Denote by $\{\mathbf{x}_i\}$, $i \in \{0, \dots, N(\beta, \mathbf{x}, r)\}$ the centers of the $N(\beta, \mathbf{x}, r)$ balls under consideration. Then, for every i , $\mathbf{x}_i \in B(\mathbf{x}, (1 - \beta)r) \cap K$.

By a simple geometric argument we see that the collection of balls $B(\mathbf{x}_i, 3\beta r)$ cover $B(\mathbf{x}, (1 - \beta)r)$. For otherwise there exists $\mathbf{y} \in B(\mathbf{x}, (1 - \beta)r)$ such that $d(\mathbf{y}, \mathbf{x}_i) \geq 3\beta r$ for every i . It follows that $B(\mathbf{y}, \beta r)$ could be added to the original collection of balls, which is a contradiction to the maximality assumption on $N(\beta, \mathbf{x}, r)$. We may assume that $\beta \leq \frac{1}{2}$ with no loss of generality, as for $\frac{1}{2} < \beta < 1$ we may choose $M \leq 2^{-\delta} \Rightarrow M\beta^{-\delta} \leq 1$. Notice also that $\delta \leq N$. And so,

$$a(1 - \beta)^\delta r^\delta \leq \mu(B(\mathbf{x}, (1 - \beta)r)) \leq N(\beta, \mathbf{x}, r)\mu(B(\mathbf{x}_i, 3\beta r)) \leq N(\beta, \mathbf{x}, r)b3^\delta \beta^\delta r^\delta.$$

$$N(\beta, \mathbf{x}, r) \geq ab^{-1}3^{-1}(1 - \beta)^\delta \beta^{-\delta} \geq ab^{-1}3^{-1}2^{-N}\beta^{-\delta}. \quad (4.16)$$

Thus Definition 4 is satisfied with $r_1 = 1$ and $M = ab^{-1}3^{-1}2^{-N}$.

□

As a consequence of Theorem 4.3 we prove the following corollary.

Corollary 4.4. *Let $\{\phi_1, \dots, \phi_k\}$ be a finite irreducible family of contracting similarity maps of \mathbb{R}^N satisfying the open set condition. Let K be its attractor and α' as in (4.15). Then for any countable collection of affine transformations $\{\Lambda_i\}_{i=0}^\infty$, with $\Lambda_i : \mathbb{R}^N \rightarrow \mathbb{R}^N$ the set*

$$\mathcal{S} = K \cap (\cap_{i=1}^\infty (\Lambda_i(\mathbf{BA})))$$

is an α -winning set for any $0 < \alpha < \frac{1}{12}\alpha'$. Furthermore, $\dim(\mathcal{S}) = \dim K$.

Proof. In view of Theorem 2.3 it suffices to prove that for each i , $K \cap \Lambda_i(\mathbf{BA})$ is α -winning. Given an affine transformation Λ and following proposition 2.1 we prove that $\mathbf{Bad}^*(\mathcal{U}, I, \rho, \Omega)$ is an α winning set on $\Omega = K$ where for every $q \in \mathbb{N}^+$

$$U_q = \left\{ \Lambda\left(\frac{\mathbf{p}}{q}\right) : \mathbf{p} \in \mathbb{Z}^N \right\}, \quad (4.17)$$

$I(q) = q$ and $\rho(I(q)) = \rho(q) = q^{\frac{-N+1}{N}}$. Following the notation of corollary 2.2 and theorem 4.1 let $\theta = \alpha\beta$ and for every $k \in \mathbb{N}^+$ let $F(k) = R^k = (\alpha\beta)^{\frac{-Nk}{N+1}}$. Define

$$U_k = \left\{ \Lambda\left(\frac{\mathbf{p}}{q}\right) : q \in \mathbb{N}^+, \mathbf{p} \in \mathbb{Z}^N \text{ and } R^{k-1} \leq q < R^k \right\}. \quad (4.18)$$

By Theorem 4.1 we get that the first condition of corollary 2.2 is satisfied by any β . As by our definition $\rho(F(k)) = (\alpha\beta)^k$, the second condition is satisfied as well. Thus $K \cap T_i(\mathbf{BA})$ is an (α, β) -winning set for every β , rendering it an α -winning set.

Furthermore, as μ is awfully friendly with the exponent of condition (iv) being $\delta = \dim K$, by proposition 3.1, followed by remark 2 we are done. □

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